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THE EFFECT OF EXTERNAL DAMPING ON THE STABILITY OF BECK'S COLUMN†

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Abstract—The stability of a cantilevered column subjected to a constant follower load in the presence of external damping is investigated. It is shown that the critical load increases with increasing damping from the value of 20·05 EI/L^2 at zero damping to the limiting value of 37·7 EI/L^2 for large damping. This behavior is in marked contrast to that of the internally damped column and also to that of conservative systems (where external damping has no effect on the critical load).

The results of this analysis, which are based on a modal approximation, are corroborated by the results obtained from the study of a standard two-degree-of-freedom model of the continuous column. Further confirmation is obtained from a computer analysis of the exact frequency equation.

THE EFFECT of damping on the stability of nonconservative systems has been the subject of much recent interest (see [1] for bibliographical information); damping can have either a stabilizing or destabilizing effect. The purpose of this note is to analyze Beck's column, a linear elastic cantilevered column subjected to a constant follower load at its free end, when external damping of a linear viscous nature is assumed to be present. Leipholz [2] and Nemat-Nasser, Prasad and Herrmann [3] have shown that the effect of this damping is not destabilizing. In this note it is shown that the effect is indeed of a stabilizing nature and the quantitative effects of this stabilization are derived. An interesting qualitative result which seems to have escaped previous investigators is also obtained: contrary to what one might expect, the critical load does not become unbounded as the damping increases.

Consider the cantilevered linear elastic column shown in Fig. 1. The equation of motion for the lateral displacement $W(X, T)$ is modeled by the partial differential equation

$$\rho\frac{\partial^2 W}{\partial T^2} + EI\frac{\partial^4 W}{\partial X^4} + P\frac{\partial^2 W}{\partial X^2} + b\frac{\partial W}{\partial T} = 0, \qquad 0 \le X \le L, T \ge 0. \tag{1}$$

The X-axis lies along the straight equilibrium shape of the column, with $X = 0$ at the built-in end and $X = L$ at the free end where the constant compressive load P is applied tangential to the column. The time is denoted by T and ρ, E, I and b are, respectively, the linear density, Young's modulus, the moment of inertia of the cross-section and the coefficient of linear viscous external damping. The boundary conditions are given by

$$W(0, T) = \frac{\partial W(0, T)}{\partial X} = \frac{\partial^2 W(L, T)}{\partial X^2} = \frac{\partial^3 W(L,T)}{\partial X^3} = 0, \qquad T \ge 0. \tag{2}$$

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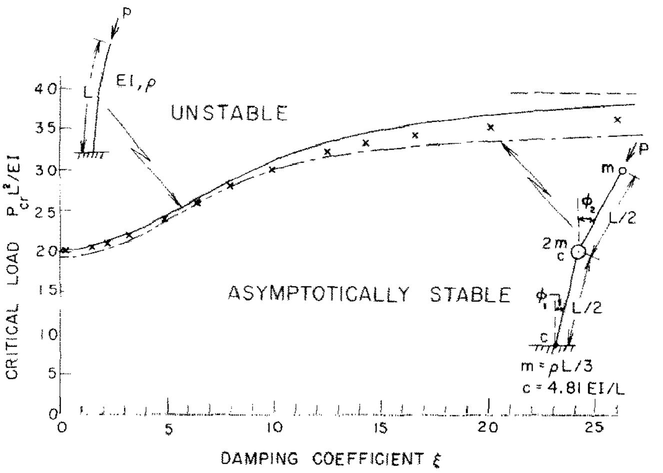

FIG. 1.

The problem is appropriately nondimensionalized by introduction of the quantities

$$p = PL^2/EI, \qquad t = \sqrt{\left(\frac{\rho L^4}{EI}\right)}T, \qquad x = \frac{X}{L} \tag{3}$$
$$w = \frac{W}{L}, \qquad \xi = \frac{1}{2}\frac{L^2}{\sqrt{(EI\rho)}}b,$$

upon which (1), (2) become

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + p\frac{\partial^2 w}{\partial x^2} + 2\xi\frac{\partial w}{\partial t} = 0, \qquad 0 \le x \le 1, \qquad t \ge 0,$$
$$w(0,t) = \frac{\partial w(0,t)}{\partial x} = \frac{\partial^2 w(1,t)}{\partial x^2} = \frac{\partial^3 w(1,t)}{\partial x^3} = 0, \qquad t \ge 0. \tag{4}$$

Consider a solution of the form

$$w(x,t) = a(t)f(x), \tag{5}$$

upon which (4) separates into the two equations

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} + 2\xi\frac{\mathrm{d}a}{\mathrm{d}t} + \lambda a = 0 \tag{6}$$

and

$$\frac{\mathrm{d}^4 f}{\mathrm{d}x^4} + p\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \lambda f = 0 \tag{7}$$

where λ is a constant. Nontrivial solutions of (7) which satisfy the boundary conditions in (4) are given by

$$f(x) = \cosh\gamma x - \cos\eta x + \mu(\gamma\sin\eta x - \eta\sinh\gamma x) \tag{8}$$

where

$$\mu = (\gamma^2 \cosh \gamma + \eta^2 \cos \eta)/(\eta\gamma^2 \sinh \gamma + \eta^2\gamma \sin \eta), \tag{9}$$

$$\gamma = \left\{ -\frac{p}{2} + \sqrt{\left(\lambda + \frac{p^2}{4}\right)} \right\}^{\frac{1}{2}}, \tag{10}$$

$$\eta = \left\{ \frac{p}{2} + \sqrt{\left(\lambda + \frac{p^2}{4}\right)} \right\}^{\frac{1}{2}}, \tag{11}$$

and λ satisfies the *frequency equation*

$$2\lambda \cos \eta \cosh \gamma + p\sqrt{(\lambda)} \sin \eta \sinh \gamma + 2\lambda + p^2 = 0. \tag{12}$$

As p increases in value from zero, the two lowest roots λ of (12), which are real, approach each other. According to Beck [4] these two roots merge when $p = 20{\cdot}05$; for higher values of p these roots form a complex conjugate pair which shall be denoted by

$$\lambda = \alpha \pm i\beta. \tag{13}$$

Substitution of (13) into (12) yields a complex equation, and setting the real and imaginary parts equal to zero leads to two transcendental equations in α and β. These equations, not listed here, are extremely complicated and untractable except for the special case $\alpha \to 0$ which yields

$$p \to 37{\cdot}7, \qquad \beta \to 191 \quad \text{as} \quad \alpha \to 0. \tag{14}$$

However, one can easily obtain approximate values for α and β by using a modal approximation. Following Deineko and Leonov [5] the approximate frequency equation

$$\lambda^2 + (13{\cdot}36p - 53{\cdot}34\pi^2)\lambda + 12{\cdot}14p^2 + 25{\cdot}75\pi^2 p + 65{\cdot}38\pi^4 = 0 \tag{15}$$

is derived with the use of the assumption

$$w(x, t) = \phi(t)y_1(x) + \psi(t)y_2(x) \tag{16}$$

where

$$y_1(x) = 5 - \cos 2\pi x - 4 \cos \pi x,$$
$$y_2(x) = 28 - \cos\frac{3\pi x}{2} - 27 \cos\frac{\pi x}{2}. \tag{17}$$

With $\lambda = \alpha \pm i\beta$, (15) yields

$$\alpha = 26{\cdot}67\pi^2 - 6{\cdot}68p,$$
$$\beta = [-32{\cdot}48p^2 + 382{\cdot}05\pi^2 p - 645{\cdot}97\pi^4]^{\frac{1}{2}} \tag{18}$$

for $p > 20{\cdot}23$.

Recall that the differential equation (6) for $a(t)$ has the solution

$$a(t) = c_1 e^{\Omega_1 t} + c_2 e^{\Omega_2 t} \tag{19}$$

where

$$\Omega_{1,2} = -\xi \pm \sqrt{(\xi^2 - \lambda)}. \tag{20}$$

If $\lambda = \alpha \pm i\beta$, then

$$\Omega_{1,2} = -\xi \pm \sqrt{(\xi^2 - \alpha - i\beta)} \quad \text{or} \quad \Omega_{1,2} = -\xi \pm \sqrt{(\xi^2 - \alpha + i\beta)}. \tag{21}$$

The column is asymptotically stable (i.e. $w(x, t) \to 0$ as $t \to \infty$) if the real part of each Ω is negative. For Ω of the form in (21), this stability condition is equivalent to the inequality

$$\alpha > \frac{\beta^2}{4\xi^2} \tag{22}$$

(see [2]). The column is unstable (i.e. $w(x, t)$ can become unbounded as $t \to \infty$) if the real part of at least one Ω is positive, that is, if

$$\alpha < \frac{\beta^2}{4\xi^2}. \tag{23}$$

The critical load p_{cr} is defined such that the column is stable if $p < p_{cr}$ and unstable if $p > p_{cr}$. It follows from (22) and (23) that p_{cr} is the minimal solution p of the equation

$$\alpha = \frac{\beta^2}{4\xi^2} \tag{24}$$

where α and β are functions of p. With the use of the approximate values of α and β given by (18), the previous equation becomes

$$26{\cdot}67\pi^2 - 6{\cdot}68p_{cr} = \frac{1}{4\xi^2}(-32{\cdot}48p_{cr}^2 + 382{\cdot}05\pi^2 p_{cr} - 645{\cdot}97\pi^4). \tag{25}$$

Solving for p_{cr} gives

$$p_{cr} \equiv \frac{P_{cr}L^2}{EI} = 5{\cdot}88\pi^2 + 0{\cdot}41\xi^2 - \sqrt{(14{\cdot}71\pi^4 + 1{\cdot}55\pi^2\xi^2 + 0{\cdot}17\xi^4)}. \tag{26}$$

This result is depicted in Fig. 1 (solid line).

A comparison of the approximate expression (26) with the exact critical load may be easily made for the extreme cases of zero damping and very large damping. For $\xi = 0$ the column is unstable whenever the frequency equation (12) has complex roots, that is, when $p > 20{\cdot}05$. The expression (26) yields $p_{cr} = 20{\cdot}23$ for $\xi \to 0$, a value 1 per cent higher than the exact critical load. For the case $\xi \to \infty$, it is seen from (24) that the exact critical load corresponds to a solution $\alpha \to 0$ of the frequency equation (12), and from (14) it then follows that this load is given by $p \to 37{\cdot}7$. According to stability condition (26), $p_{cr} \to 39{\cdot}4$ as $\xi \to \infty$ (see the horizontal dashed line in Fig. 1), so that the approximate critical load p_{cr} is $4\frac{1}{2}$ per cent higher than the exact value as $\xi \to \infty$.

From these results, which are summarized in Fig. 1, it is concluded that external damping has a stabilizing effect on Beck's column. The critical load increases monotonically with increasing damping but does not become unbounded; in fact the lower and upper bounds on the critical load are $p = 20{\cdot}05$, corresponding to zero damping, and $p = 37{\cdot}7$ for very large damping.

Since these results were obtained by means of an approximate modal analysis, it seems appropriate to corroborate them through another type of approximation. An often used model for the cantilevered column is the inverted double pendulum of Fig. 1. This model and several variants have been recently analyzed by Herrmann and his students.

The bar lengths are both taken as $L/2$, the masses are $2m$ and m, and the restoring moments at the joints are $c\phi_1$ and $c(\phi_2 - \phi_1)$. Since it is assumed that the damping is proportional to the velocity along the double pendulum with proportionality constant b, the linearized dissipation function is easily computed as $D = (bL^3/48)(4\dot{\phi}_1^2 + 3\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2)$. The equations of motion of the model then are obtained, after appropriate linearization, as

$$\frac{3}{4}mL^2\ddot{\phi}_1 + \frac{1}{6}bL^3\dot{\phi}_1 - \left(P\frac{L}{2} - 2c\right)\phi_1 + \frac{1}{4}mL^2\ddot{\phi}_2 + \frac{1}{16}bL^3\dot{\phi}_2 + \left(\frac{PL}{2} - c\right)\phi_2 = 0$$

$$\frac{1}{4}mL^2\ddot{\phi}_1 + \frac{1}{16}bL^3\dot{\phi}_1 - c\phi_1 + \frac{1}{4}mL^2\ddot{\phi}_2 + \frac{1}{24}bL^3\dot{\phi}_2 + c\phi_2 = 0. \tag{27}$$

Assuming a solution of the form $\phi_i = A_i e^{\omega t}$ and letting

$$\Omega = \frac{L}{2}\sqrt{\left(\frac{m}{c}\right)}\omega, \qquad F = \frac{PL}{2c}, \qquad B = \frac{bL^2}{4\sqrt{(cm)}} \tag{28}$$

we obtain through a simple application of the Routh–Hurwitz conditions that for stability it is necessary and sufficient that

$$\tfrac{5}{8}F^2 - \left(\tfrac{13}{3} + \tfrac{35}{216}B^2\right)F + \tfrac{37}{6} + \tfrac{7}{12}B^2 > 0, \qquad F < 3{\cdot}6, \qquad F < 3{\cdot}5 + \tfrac{7}{72}B^2, \tag{29}$$

and from this it immediately follows that the critical load is governed by

$$F_{cr} = \tfrac{4}{5}\left\{\tfrac{13}{3} + \tfrac{35}{216}B^2 - \sqrt{\left[\left(\tfrac{13}{3} + \tfrac{35}{216}B^2\right)^2 - \tfrac{5}{2}\left(\tfrac{37}{6} + \tfrac{7}{12}B^2\right)\right]}\right\}. \tag{30}$$

At this juncture, it is important to relate the parameters of the model to those of the continuous column. Clearly $m = \rho L/3$; the restoring moment proportionality constant c is determined by the condition that the critical load of the model and of the column be the same for zero damping, that is, [4,6]

$$20{\cdot}05\frac{EI}{L^2} = 2\left(2{\cdot}086\frac{c}{L}\right) \tag{31}$$

which yields $c = 4{\cdot}81\, EI/L$. Hence, in terms of the variables of the original column, (30) becomes

$$\frac{P_{cr}L^2}{EI} = 33{\cdot}4 + 0{\cdot}195\xi^2 - \sqrt{(202 - 0{\cdot}503\xi^2 + 0{\cdot}038\xi^4)}. \tag{32}$$

This result is shown in Fig. 1. It is similar both qualitatively and quantitatively to the result obtained through the modal approximation.

In order to further verify the behavior of the critical load of the cantilevered continuous column, a computer analysis of the exact frequency equation (12) was carried out for several values of p. The following values of α and β were obtained:

p	20·06	20·50	21·00	22·00	24·00	26·00	28·00
α	121·3	118·4	115·1	108·5	95·25	81·85	68·31
β	4·557	32·04	46·48	66·33	93·64	114·1	130·9
p	30·00	32·00	33·00	34·00	35·00	36·00	37·00
α	54·62	40·74	33·73	26·66	19·54	12·35	5·090
β	145·6	158·7	164·8	170·8	176·5	182·1	187·5

The stability condition (24) then yields corresponding values of ξ for each p, and the resulting critical loads are shown by x's in Fig. 1. As expected from the discussion following (26), these values are from 1 to $4\frac{1}{2}$ per cent lower than the critical loads obtained by the modal approximation.

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Абстракт—Исследуется устойчивость консольной колонны, подверженной действию постоянной, следящей нагрузки, при наличии внешнего демпфирования. Показано, что критическая нагрузка увеличается с ростом демпфирования от значения 20,05 EI/L^2, для нулевого демпфирования, до предельного значения составляющего—37,7 EI/L^2, для значительного демпфирования. Такое поведение разнится значительно по сравнению с внутренным демпфированием колонны, а также по сравнению с консервативными системами 'у которых внешнее демпфирование не вызывает никакого эффекта на критическую нагрузку'.

Результаты этого анализа, основанные на модальном приближении, подтверждаются результатами полученными при исследовании стандартной модели непрерывной колонны, с двумя степенями свободы. Дальнейшее подтверждение получается из анализа, проведенного на вычислительной машине, для точного уравнения частоты.